

# PARTITIONS WITH PARTS IN A FINITE SET AND THE NON-INTERSECTING CIRCLES PROBLEM

MADJID MIRZAVAZIRI AND DANIEL YAQUBI

ABSTRACT. Let  $n$  be a non-negative integer and  $A = \{a_1, \dots, a_k\}$  be a multi-set with  $k$  not necessarily distinct members, where  $a_1 \leq \dots \leq a_k$ . We denote by  $\Delta(n, A)$  the number of ways to partition  $n$  as the form  $a_1x_1 + \dots + a_kx_k$ , where  $x_i$ 's are distinct positive integers and  $x_i < x_{i+1}$  whenever  $a_i = a_{i+1}$ . We give a recursive formula for  $\Delta(n, A)$  and some explicit formulas for some special cases. Using this notion we solve the non-intersecting circles problem which asks to evaluate the number of ways to draw  $n$  non-intersecting circles in a plane regardless to their sizes. The latter also enumerates the number of unlabelled rooted tree with  $n + 1$  vertices.

## 1. INTRODUCTION

Let  $n$  be a positive integer. A partition of  $n$  is a representation of  $n$  as a sum of positive integers. The *partition function*, denoted by  $\Pi$ , enumerates the partition of the positive integer  $n$ . In the case of partitioning  $n$  into a specified number of parts, say  $k$ , the terminology *partitions of  $n$  into  $k$  parts* is used. Since the order of parts in a partition does not count, they are registered in a decreasing order of magnitude. Thus, the partition of  $n$  into  $k$  parts is the number of solutions in positive integers of the linear equation

$$n = x_1 + x_2 + \dots + x_k, \quad x_1 \geq x_2 \geq \dots \geq x_k \geq 1$$

which is denoted by  $\Pi(n, k)$ . For example,  $7 = 5 + 1 + 1 = 4 + 2 + 1 = 3 + 3 + 1 = 3 + 2 + 2$ , so that  $\Pi(7, 3) = 4$ . We see that  $\Pi(n, k)$  equals the number of solutions of

$$n - k = y_1 + y_2 + \dots + y_k, \quad y_1 \geq y_2 \geq \dots \geq y_k \geq 0.$$

If exactly  $s$  of the integers  $y_i$  are positive, there are  $\Pi(n - k, s)$  solutions  $(y_1, y_2, \dots, y_k)$ . Therefore we have the recursive formula  $\Pi(n, k) = \sum_{s=0}^k \Pi(n - k, s)$ . Since we have the trivial initial conditions  $\Pi(n, n) = \Pi(0, 0) = 1$  and  $\Pi(n, k) = 0$  for  $n < k$ , we can

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recursively calculate the numbers  $\Pi(n, k)$ . Clearly  $\Pi(n, 1) = 1$  and  $\Pi(n, 2) = \lfloor \frac{n}{2} \rfloor$ . It can be easily shown that  $\Pi(n)$  is the number of solutions of

$$n = x_1 + x_2 + \dots + x_n, \quad x_1 \geq x_2 \geq \dots \geq x_k \geq 0.$$

This can also be formulated as the number of solutions of  $n = y_1 + 2y_2 + \dots + ny_n$ , where  $y_i \geq 0$ , for  $1 \leq i \leq n$ .

A multi-set  $A$  is a pair  $(S, m)$ , where  $S$  is a set and  $m$  is a function from  $S$  to  $\mathbb{N}$ . The set  $S$  is called the *set of underlying elements* of  $A$  and is denoted by  $S(A)$ . For each  $a \in S$  the multiplicity of  $a$  is given by  $m(a)$ . A multi-set  $A$  is called an  $\ell$ -multi-set if  $\sum_{a \in A} m(a) = \ell$  for some  $\ell \in \mathbb{N}$ . A formal definition of a multi-set can be found in [1]. Now, let  $A = \{a_1, a_2, \dots, a_k\}$  be a multi-set of  $k$  positive integers. A *partition* of  $n$  with respect to the multi-set  $A$  is a representation of  $n$  as a sum of elements of  $A$ . More on integer partitions can be found in [2, 3, 4, 5, 6, 9, 8] and [11].

## 2. THE RESULTS

**Definition 2.1.** Let  $n$  be a non-negative integer and  $A = \{a_1, \dots, a_k\}$  be a multi-set with  $k$  not necessarily distinct members. We denote by  $D(n, A)$  the number of ways to partition  $n$  as the form  $a_1x_1 + \dots + a_kx_k$ , where  $x_i$ 's are positive integers and  $x_i \leq x_{i+1}$  whenever  $a_i = a_{i+1}$ . The number of ways to partition  $n$  as the form  $a_1x_1 + \dots + a_kx_k$ , where  $x_i$ 's are non-negative integers and  $x_i \leq x_{i+1}$  whenever  $a_i = a_{i+1}$ , is also denoted by  $D_0(n, A)$ . The numbers  $D(n, A)$  and  $D_0(n, A)$  are called *the natural partition number* and *the arithmetic partition number of  $n$  with respect to  $A$* , respectively.

**Proposition 2.2.** Let  $n$  be a non-negative integer and  $A$  be a multi-set with the multiplicity mapping  $m$ . Then for each  $a \in A$

$$D(n, A) = \sum_{\substack{0 \leq \ell \leq m(a) \\ am(a) \leq t}} D(n - am(a), A \setminus aI_\ell), \quad (2.1)$$

where  $D(0, \emptyset) = 1$ . In particular,  $D(n, I_k) = \Pi(n, k)$ . Furthermore,

$$D_0(n, A) = D(n + \sigma(A), A).$$

*Proof.* Let  $A = \{a_1, \dots, a_k\}$ . We have  $m(a)$  occurrence of  $a$  in the equation  $n = a_1x_1 + \dots + a_kx_k$ . Let  $x_{i+1}, \dots, x_{i+m(a)}$  have coefficients  $a$  in the equation,  $x_{i+1} = \dots = x_{i+\ell} = 1$  and  $x_{i+\ell+1} > 1$ , where  $\ell = 0, 1, \dots, m(a)$ . If we subtract  $am(a)$  from the both sides of the equation  $n = a_1x_1 + \dots + a_kx_k$  then we reach into the equation  $n - am(a) =$

$a_1x_1 + \dots + a_ix_i + a_{i+\ell+1}x_{i+\ell+1} + \dots + a_kx_k$ . The natural partition number of the latter equation is  $D(n - am(a), A \setminus aI_\ell)$ . The other parts of the above assertion is clear.  $\square$

**Corollary 2.3.** *Let  $n$  be a positive integer. Then  $D(n, \{1, 2\}) = \lfloor \frac{n-1}{2} \rfloor$ ,  $D(n, \{1, 2, 2\}) = \lfloor \frac{n-1}{4} \rfloor (\lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n+3}{4} \rfloor)$  and  $D(n, \{1, 1, 2\}) = \lfloor \frac{3}{2} \lfloor \frac{n-1}{3} \rfloor + \frac{1}{2} \rfloor (\lfloor \frac{n-1}{2} \rfloor - \frac{1}{2} \lfloor \frac{3}{2} \lfloor \frac{n+2}{3} \rfloor \rfloor + \frac{1+(-1)^n}{2})$ .*

*Proof.* Let  $n = 2k + r$ , where  $r = 1, 2$ . Using Proposition 2.2 we can write

$$\begin{aligned} D(n, \{1, 2\}) &= D(n-2, \{1, 2\}) + D(n-2, \{1\}) = D(n-2, \{1, 2\}) + 1 \\ &= D(n-4, \{1, 2\}) + D(n-2, \{1\}) + 1 = D(n-4, \{1, 2\}) + 2 \\ &= D(n-6, \{1, 2\}) + 3 = \dots = D(n-2k, \{1, 2\}) + k \\ &= 0 + k = \lfloor \frac{n-1}{2} \rfloor. \end{aligned}$$

For the second assertion, let  $n = 4k + r$ , where  $r = 1, 2, 3, 4$ . then

$$\begin{aligned} D(n, \{1, 2, 2\}) &= D(n-4, \{1, 2, 2\}) + D(n-4, \{1, 2\}) + D(n-4, \{1\}) \\ &= D(n-4, \{1, 2, 2\}) + \lfloor \frac{n-5}{2} \rfloor + 1 \\ &= D(n-8, \{1, 2, 2\}) + \lfloor \frac{n-7}{2} \rfloor + \lfloor \frac{n-3}{2} \rfloor \\ &= \dots \\ &= D(n-4k, \{1, 2, 2\}) + \sum_{i=1}^k \lfloor \frac{n-(4i-1)}{2} \rfloor \\ &= D(r, \{1, 2, 2\}) + \sum_{i=1}^k \lfloor \frac{n-(4i-1)}{2} \rfloor \\ &= 0 + \sum_{i=1}^k \lfloor \frac{n-(4i-1)}{2} \rfloor \\ &= k \lfloor \frac{n+1}{2} \rfloor - k(k+1) \\ &= \lfloor \frac{n-1}{4} \rfloor (\lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n+3}{4} \rfloor). \end{aligned}$$

Now, let  $n = 3k + r$  where  $r = 1, 2, 3$ , then

$$\begin{aligned}
D(n, \{1, 1, 2\}) &= D(n-2, \{1, 1, 2\}) + D(n-2, \{1, 2\}) + D(n-2, \{2\}) \\
&= D(n-2, \{1, 1, 2\}) + \lfloor \frac{n-3}{2} \rfloor + \frac{1+(-1)^n}{2} \\
&= D(n-4, \{1, 1, 2\}) + \lfloor \frac{n-5}{2} \rfloor + \lfloor \frac{n-3}{2} \rfloor + 2(\frac{1+(-1)^n}{2}) \\
&= \dots \\
&= D(n-4(\lfloor \frac{3k+1}{2} \rfloor), \{1, 1, 2\}) \\
&\quad + \sum_{i=1}^{\lfloor \frac{3k+1}{2} \rfloor} \lfloor \frac{(n-2i)-1}{2} \rfloor + \lfloor \frac{3k+1}{2} \rfloor (\frac{1+(-1)^n}{2}) \\
&= 0 + \sum_{i=1}^{\lfloor \frac{3k+1}{2} \rfloor} \lfloor \frac{(n-2i)-1}{2} \rfloor + k(\frac{1+(-1)^n}{2}) \\
&= \lfloor \frac{3k+1}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor - \frac{\lfloor \frac{3k+1}{2} \rfloor (\lfloor \frac{3k+1}{2} \rfloor + 1)}{2} + \lfloor \frac{3k+1}{2} \rfloor (\frac{1+(-1)^n}{2}).
\end{aligned}$$

It is enough to note that  $k = \lfloor \frac{n-1}{3} \rfloor$ . □

**Example 2.4.** We evaluate  $D(17, \{1, 2, 2, 3\})$  and  $D_0(17, \{1, 2, 2, 3\})$ . Using Proposition 2.2 and Corollary 2.3 we can write

$$\begin{aligned}
&D(17, \{1, 2, 2, 3\}) \\
&= D(14, \{1, 2, 2, 3\}) + D(14, \{1, 2, 2\}) \\
&= D(11, \{1, 2, 2, 3\}) + D(11, \{1, 2, 2\}) + 9 \\
&= D(8, \{1, 2, 2, 3\}) + D(8, \{1, 2, 2\}) + 6 + 9 \\
&= 1 + 2 + 6 + 9 \\
&= 18.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& D_0(17, \{1, 2, 2, 3\}) \\
&= D(17 + 8, \{1, 2, 2, 3\}) \\
&= D(22, \{1, 2, 2, 3\}) + D(22, \{1, 2, 2\}) \\
&= D(19, \{1, 2, 2, 3\}) + D(19, \{1, 2, 2\}) + 25 \\
&= D(16, \{1, 2, 2, 3\}) + D(16, \{1, 2, 2\}) + 20 + 25 \\
&= D(13, \{1, 2, 2, 3\}) + D(13, \{1, 2, 2\}) + 12 + 20 + 25 \\
&= D(10, \{1, 2, 2, 3\}) + D(10, \{1, 2, 2\}) + 9 + 12 + 20 + 25 \\
&= D(7, \{1, 2, 2, 3\}) + D(7, \{1, 2, 2\}) + 4 + 9 + 12 + 20 + 25 \\
&= 0 + 2 + 4 + 9 + 12 + 20 + 25 = 72.
\end{aligned}$$

**Proposition 2.5.** *Let  $n$  be a positive integer. Then*

$$D\left(\frac{n(n+3)}{2}, \{1, 2, \dots, n\}\right) = D(2n, \{1, 1, \dots, 1\}).$$

**Lemma 2.6.** *Let  $n$  be a positive integer and  $A = \{a_1, a_2\}$ . If  $a_1 = a_2$  then  $D(n, \{a_1, a_2\}) = \lfloor \frac{n}{2a_1} \rfloor$ , and if  $a_1 \neq a_2$  then  $D(n, \{a_1, a_2\}) = \lfloor \frac{n-1}{a_1 a_2} \rfloor$ .*

**Corollary 2.7.** *Let  $n$  be a positive integer. Then*

$$\begin{aligned}
D(n, \{1, 2, 3\}) &= \lfloor \frac{n-4}{6} \rfloor \left( \frac{2\lfloor \frac{n-3}{2} \rfloor - \lfloor \frac{n+2}{6} \rfloor}{2} \right) - \frac{\lfloor \frac{2\lfloor \frac{n-4}{2} \rfloor}{3} \rfloor \lfloor \frac{2\lfloor \frac{n-4}{2} \rfloor - 3}{3} \rfloor}{2} \\
&\quad + \lfloor \frac{2\lfloor \frac{n-4}{2} \rfloor + 3}{6} \rfloor \left( \frac{2\lfloor \frac{n-2}{2} \rfloor - \lfloor \frac{2\lfloor \frac{n-4}{2} \rfloor + 9}{6} \rfloor}{2} \right).
\end{aligned}$$

*Proof.* By Proposition 2.5, Lemma 2.6 and the last part of Proposition 2.2, we have  $D_0(n-6, \{1, 2, 3\}) = D_0(n-6, \{1, 1, 1\})$ . Thus  $D(n, \{1, 2, 3\}) = D(n-3, \{1, 1, 1\})$ . Put

$n_1 = n - 3$  and let  $n_1 = 2k + r$ , where  $r = 1, 2$ . Hence

$$\begin{aligned}
D(n_1, \{1, 1, 1\}) &= D(n_1 - 3, \{1, 1, 1\}) + D(n_1 - 3, \{1, 1\}) + D(n_1 - 3, \{1\}) \\
&= D(n_1 - 3, \{1, 1, 1\}) + \lfloor \frac{n_1 - 3}{2} \rfloor + 1 \\
&= D(n_1 - 6, \{1, 1, 1\}) + \lfloor \frac{n_1 - 6}{2} \rfloor + \lfloor \frac{n_1 - 3}{2} \rfloor + 1 + 1 \\
&= \dots \\
&= D(n_1 - 3(\lfloor \frac{2k}{3} \rfloor), \{1, 1, 1\}) + \sum_{i=1}^{\lfloor \frac{2k}{3} \rfloor} \lfloor \frac{n_1 - 3i}{2} \rfloor + \lfloor \frac{2k}{3} \rfloor \\
&= D(r, \{1, 1, 1\}) + \sum_{i=1}^{\lfloor \frac{2k}{3} \rfloor} \lfloor \frac{n_1 - 3i}{2} \rfloor + \lfloor \frac{2k}{3} \rfloor \\
&= 0 + \sum_{i=1}^{\lfloor \frac{2k}{3} \rfloor} \lfloor \frac{n_1 - i}{2} \rfloor - \frac{\lfloor \frac{2k}{3} \rfloor \lfloor \frac{2k-3}{3} \rfloor}{2} \\
&= \sum_{i=1}^{\lfloor \frac{2k}{6} \rfloor} (\lfloor \frac{n_1}{2} \rfloor - i) + \sum_{i=1}^{\lfloor \frac{2k+3}{6} \rfloor} (\lfloor \frac{n_1+1}{2} \rfloor - i) - \frac{\lfloor \frac{2k}{3} \rfloor \lfloor \frac{2k-3}{3} \rfloor}{2} \\
&= \lfloor \frac{k}{3} \rfloor (\frac{2\lfloor \frac{n_1}{2} \rfloor - \lfloor \frac{k+3}{3} \rfloor}{2}) + \lfloor \frac{2k+3}{6} \rfloor (\frac{2\lfloor \frac{n_1+1}{2} \rfloor - \lfloor \frac{2k+9}{6} \rfloor}{2}) - \frac{\lfloor \frac{2k}{3} \rfloor \lfloor \frac{2k-3}{3} \rfloor}{2}.
\end{aligned}$$

Now note that  $k = \lfloor \frac{n_1-1}{2} \rfloor$  and  $n_1 = n - 3$ . □

**Definition 2.8.** Let  $n$  be a non-negative integer and  $A = \{a_1, \dots, a_k\}$  be a multi-set with  $k$  not necessarily distinct members, where  $a_1 \leq \dots \leq a_k$ . We denote by  $\Delta(n, A)$  the number of ways to partition  $n$  as the form  $a_1x_1 + \dots + a_kx_k$ , where  $x_i$ 's are distinct positive integers and  $x_i < x_{i+1}$  whenever  $a_i = a_{i+1}$ . The number of ways to partition  $n$  as the form  $a_1x_1 + \dots + a_kx_k$ , where  $x_i$ 's are distinct non-negative integers and  $x_i < x_{i+1}$  whenever  $a_i = a_{i+1}$ , is also denoted by  $\Delta_0(n, A)$ . The numbers  $\Delta(n, A)$  and  $\Delta_0(n, A)$  are called the *natural distinct partition number* and the *arithmetic distinct partition number of  $n$  with respect to  $A$* .

P. A. Macmahon [7] was the first mathematician interested in set partitions. Also, E. Deutsch studied the number of partitions of  $n$  into exactly two odd size of parts (see A117955 of [10]) and the number of partitions of  $n$  into exactly two sizes parts, one odd and one even (see A117756 of [10]).

**Proposition 2.9.** *Let  $n$  be a non-negative integer and  $A$  be a multi-set with the background set  $S(A) = \{b_1, \dots, b_\ell\}$ . Then*

$$\Delta(n, A) = \Delta(n - \sigma(A), A) + \sum_{i=1}^{\ell} \Delta(n - \sigma(A), A \setminus \{b_i\}). \quad (2.2)$$

Moreover, if  $A = \{a_1, \dots, a_k\}$  with  $a_1 \leq \dots \leq a_k$  then  $\Delta(n, A) = 0$  when  $n < \sum_{i=1}^k (k + 1 - i)a_i$ . Furthermore,

$$\Delta_0(n, A) = \Delta(n + \sigma(A), A).$$

*Proof.* At most one of  $x_i$ 's can be 1. If there is no  $x_i$  with  $x_i = 1$  then we can write  $n - \sigma(A) = a_1(x_1 - 1) + \dots + a_k(x_k - 1)$  and there are  $\Delta(n - \sigma(A), A)$  solutions for this equation under the required conditions. Moreover, if  $x_j = 1$  for some  $j$ , then other  $x_i$ 's are greater than 1 and thus we can write  $n - \sigma(A) = a_1(x_1 - 1) + \dots + a_{j-1}(x_{j-1} - 1) + a_{j+1}(x_{j+1} - 1) + \dots + a_k(x_k - 1)$ . There are  $\Delta(n - \sigma(A), A \setminus \{b_j\})$  solutions for the latter equation, where  $b_j = a_j$ . The other parts are obvious.  $\square$

**Corollary 2.10.** *Let  $n$  be a positive integer. Then  $\Delta(n, \{1, 1\}) = \lfloor \frac{n-1}{2} \rfloor$  and  $\Delta(n, \{1, 2\}) = \lfloor \frac{n-1}{3} \rfloor + \lfloor \frac{n-1}{6} \rfloor$ .*

*Proof.* Let  $n = 2k + r$ , where  $r = 1, 2$ . Using Proposition 2.9 we can write

$$\begin{aligned} \Delta(n, \{1, 1\}) &= \Delta(n - 2, \{1, 1\}) + \Delta(n - 2, \{1\}) \\ &= \Delta(n - 2, \{1\}) + 1 \\ &= \Delta(n - 4, \{1, 1\}) + \Delta(n - 4, \{1\}) + 1 \\ &= \dots \\ &= \Delta(n - 2k, \{1, 1\}) + k \\ &= 0 + k \\ &= \lfloor \frac{n-1}{2} \rfloor. \end{aligned}$$

Now let  $n = 3k + r$ , where  $r = 1, 2, 3$ . Thus

$$\begin{aligned}
\Delta(n, \{1, 2\}) &= \Delta(n - 3\{1, 2\}) + \Delta(n - 3, \{1\}) + \Delta(n - 3, \{2\}) \\
&= \Delta(n - 3, \{1, 2\}) + \Delta(n - 3, \{2\}) + 1 \\
&= \Delta(n - 6, \{1, 2\}) + \Delta(n - 6, \{1\}) + \Delta(n - 6, \{2\}) + \Delta(n - 3, \{2\}) + 1 \\
&= \Delta(n - 6, \{1, 2\}) + \Delta(n - 6, \{2\}) + \Delta(n - 3, \{2\}) + 2 \\
&= \dots \\
&= \Delta(n - 3k, \{1, 2\}) + \sum_{i=1}^{\lfloor \frac{n-1}{3} \rfloor} \Delta(n - 3i, \{2\}) + k \\
&= 0 + \sum_{i=1}^{\lfloor \frac{n-1}{3} \rfloor} \Delta(n - 3i, \{2\}) + \lfloor \frac{n-1}{3} \rfloor.
\end{aligned}$$

If  $n = 3k$  then  $k - i$  is even and so

$$\sum_{i=1}^{\lfloor \frac{n-1}{3} \rfloor} \Delta(n - 3i, \{2\}) + \lfloor \frac{n-1}{3} \rfloor = \lfloor \frac{n-1}{6} \rfloor + \lfloor \frac{n-1}{3} \rfloor.$$

Similarly, we have the result for the cases  $n = 3k + 1$  and  $n = 3k + 2$ . □

**Example 2.11.** We evaluate  $\Delta(18, \{1, 2, 2, 3\})$ . Regarding to Proposition 2.9 and corollary 2.10 we have

$$\begin{aligned}
&\Delta(18, \{1, 2, 2, 3\}) \\
&= \Delta(10, \{1, 2, 2, 3\}) + \Delta(10, \{2, 2, 3\}) + \Delta(10, \{1, 2, 3\}) + \Delta(10, \{1, 2, 2\}) \\
&= 0 + 0 + \Delta(4, \{1, 2, 3\}) + \Delta(4, \{2, 3\}) + \Delta(4, \{1, 3\}) + \Delta(4, \{1, 2\}) \\
&\quad + \Delta(5, \{1, 2, 2\}) + \Delta(5, \{2, 2\}) + \Delta(5, \{1, 2\}) \\
&= 0 + 0 + 0 + 0 + 0 + 1 + 0 + 0 + 2 \\
&= 3.
\end{aligned}$$

The 3 solutions are

$$\begin{aligned}
18 &= 1 \times \mathbf{3} + 2 \times \mathbf{2} + 2 \times \mathbf{4} + 3 \times \mathbf{1} \\
&= 1 \times \mathbf{5} + 2 \times \mathbf{2} + 2 \times \mathbf{3} + 3 \times \mathbf{1} \\
&= 1 \times \mathbf{4} + 2 \times \mathbf{1} + 2 \times \mathbf{3} + 3 \times \mathbf{2}.
\end{aligned}$$



## 3. APPLICATION

To solve the non-intersecting circles problem, let us assume the following notations.

Let  $n$  be a positive integer. We denote the set of all multi-sets  $A = \{a_1, \dots, a_k\}$  such that there are distinct positive integers  $x_1, \dots, x_k$  with  $n = a_1x_1 + \dots + a_kx_k$ , where  $x_i < x_{i+1}$  whenever  $a_i = a_{i+1}$ , by  $\mathcal{A}_{n,k}$ . Recall that for an  $A \in \mathcal{A}_{n,k}$  there are  $\Delta(n, A)$  solutions  $(x_1, \dots, x_k)$  satisfying the above condition. We denote the set of these  $(x_1, \dots, x_k)$  by  $\mathcal{X}_A$ .

*Remark 3.1.* We also recall that the number of  $(n_1, \dots, n_r)$  with  $1 \leq n_1 \leq \dots \leq n_r \leq s$  is equal to  $\sum_{i=1}^k \binom{r-1}{i-1} \binom{s}{i} = \sum_{i=1}^k \binom{r-1}{r-i} \binom{s}{i} = \binom{r+s-1}{r}$ .

The non-intersecting circles problem asks to evaluate the number of ways to draw  $n$  non-intersecting circles in a plane regardless to their sizes. For example, if we use the symbol  $( )$  for a circle then there are 4 formats for 3 circles

$$( ) ( ) ( ), (( ) ( )), (( )) ( ), ((( )))$$

and 9 formats for 4 circles

$$\begin{aligned} & ( ) ( ) ( ) ( ), (( ) ( ) ( )), (( ) ( )) ( ), ((( ) ( ))), \\ & (( )) ( ) ( ), ((( )) ( )), ((( )) ( )) ( ), ((( ( ))) ), \\ & (( )) (( )). \end{aligned}$$

If we denote the answer by  $C_n$  then we can see that  $C_0 = C_1 = 1, C_2 = 2, C_3 = 4, C_4 = 9$  and  $C_5 = 20$ .

**Theorem 3.2.** *Let  $C_n$  be the number of ways to draw  $n$  non-intersecting circles in a plane regardless to their sizes. Then*

$$C_n = \sum_{k=1}^{\lfloor \sqrt{2n} \rfloor} \sum_{A=\{a_1, \dots, a_k\} \in \mathcal{A}_{n,k}} \sum_{(x_1, \dots, x_k) \in \mathcal{X}_A} \prod_{i=1}^k \binom{C_{x_{i-1}} + a_i - 1}{a_i}.$$

*Proof.* Given  $n$ , let we draw our circles in  $\ell$  parts with  $y_i$  circles in  $i$ -th part. We can assume that  $y_1 \leq \dots \leq y_\ell$ . Thus  $n = y_1 + \dots + y_\ell$ . We can rewrite it as the form  $n = a_1x_1 + \dots + a_kx_k$  such that  $x_i < x_{i+1}$  whenever  $a_i = a_{i+1}$ . This shows that we have  $a_i$  parts with  $x_i$  circles of the form  $(x_i - 1)$  where  $( )$  denotes a circle containing  $x_i - 1$  circles. We can form the  $a_i$  parts of the form  $(x_i - 1)$  in  $\binom{C_{x_{i-1}} + a_i - 1}{a_i}$  ways. The latter

is true since we may put  $r = a_i$  and  $s = C_{x_i-1}$  in Remark 3.1. Note that a single form  $(x_i - 1)$  can be drawn in  $C_{x_i-1}$  ways.

Now notice the fact that the maximum of  $k$  occurs when  $a_1 = \dots = a_k = 1$ . Since we have  $1 \leq x_1 < \dots < x_k$  in this case, we can therefore deduce that  $\frac{k(k+1)}{2} \leq n$ . Thus the maximum value of  $k$  is  $\lfloor \sqrt{2n} \rfloor$ .  $\square$

**Example 3.3.** For  $n = 6$  we have

$$\begin{aligned}\mathcal{A}_{6,1} &= \{\{1\}, \{2\}, \{3\}, \{6\}\} \\ \mathcal{A}_{6,2} &= \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 2\}\} \\ \mathcal{A}_{6,3} &= \{\{1, 1, 1\}\}\end{aligned}$$

We can therefore write

$$\begin{aligned}6 &= 1 \times 6 = 2 \times 3 = 3 \times 2 = 6 \times 1 \\ &= 1 \times 1 + 1 \times 5 = 1 \times 2 + 1 \times 4 \\ &= 1 \times 4 + 2 \times 1 \\ &= 1 \times 3 + 3 \times 2 \\ &= 1 \times 2 + 4 \times 1 \\ &= 2 \times 1 + 2 \times 2 \\ &= 1 \times 1 + 1 \times 2 + 1 \times 3.\end{aligned}$$

Thus

$$\begin{aligned}C_6 &= \binom{C_5}{1} + \binom{C_2+1}{2} + \binom{C_1+2}{3} + \binom{C_0+5}{6} \\ &\quad + \binom{C_0}{1} \binom{C_4}{1} + \binom{C_1}{1} \binom{C_3}{1} \\ &\quad + \binom{C_3}{1} \binom{C_0+1}{2} + \binom{C_2}{1} \binom{C_1+2}{3} + \binom{C_1}{1} \binom{C_0+3}{4} + \binom{C_0+1}{2} \binom{C_1+1}{2} \\ &\quad + \binom{C_0}{1} \binom{C_1}{1} \binom{C_2}{1} \\ &= 20 + 3 + 1 + 1 + 9 + 4 + 4 + 2 + 1 + 1 + 2 \\ &= 48.\end{aligned}$$

**Corollary 3.4.** *Let  $n$  be a positive integer. Then  $C_n$  is the number of unlabelled rooted tree with  $n + 1$  vertices.*

*Proof.* There is a one to one correspondence between  $n$  non-intersecting circles and an unlabelled rooted tree with  $n + 1$  vertices. It is enough to draw a circle for each non-root vertex and put a circle inside another one if the second one is the parent of the first one.  $\square$

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DEPARTMENT OF PURE MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P. O. Box 1159, MASHHAD 91775, IRAN.

*E-mail address:* mirzavaziri@gmail.com and daniel\_yaqubi@yahoo.es